In-Plane Vibration of Spinning Disks

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IN an earlier investigation,[1] the effect of "rigid body" rotation on the vibration of elastic bodies was discussed in general terms. It was shown that the rotation couples the transverse- and longitudinal-type vibrations and that it tends to increase the natural frequency of the longitudinal vibration while decreasing the natural frequency of the transverse vibration. Analytically, this effect is produced primarily by terms commonly referred to as Coriolis acceleration. Recently, Di Taranto and Lessen[2] have studied this effect in rotating cylindrical shells. The object here is to investigate this effect in the "in-plane" vibration of rotating circular disks.†

Governing Equations

Following Refs. 3 and 1 and considering only axially symmetric motion, the governing equations may be written

$$\partial^2 u/\partial r^2 + \partial(u/r)/\partial r = [\zeta(1 - \nu^2)/E][\partial^2 u/\partial t^2 - 2\Omega\partial v/\partial t] \quad (1)$$

and

$$\partial^2 v/\partial r^2 + \partial(v/r)/\partial r = [2\zeta(1 + \nu)/E][\partial^2 v/\partial t^2 + 2\Omega\partial u/\partial t] \quad (2)$$

where u and v are the radial and tangential displacements, r is the radial coordinate, and Ω is the angular speed of the disk, which is taken to be rotating about its axis. E, ν, and ζ are physical constants representing, respectively, Young's modulus, Poisson's ratio, and the mass density. In developing these equations, the coordinate system is taken to be rotating with speed Ω about the disk axis, and the partial time derivatives are computed with respect to this rotating system. If the disk is traction free, the boundary conditions take the form

$$\partial u/\partial r + (\nu/r)u = 0 \qquad \text{at } r = R \quad (3)$$

and

$$\partial v/\partial r - v/r = 0 \qquad \text{at } r = R \quad (4)$$

where R is the disk radius.

Vibrations

For oscillatory motion, the solutions of Eqs. (1) and (2) are assumed to be

$$u = U \cos\omega t \quad (5)$$

and

$$v = V \sin\omega t \quad (6)$$

where U and V are functions of r only, and ω is the natural frequency. Substituting Eqs. (5) and (6) into (1) and (2) and introducing the dimensionless parameter ξ as r/R results in the equations

$$d^2U/d\xi^2 + d(U/\xi)/d\xi + [\zeta R^2(1 - \nu^2)/E][\omega^2 U + 2\Omega\omega V] = 0 \quad (7)$$

and

$$d^2V/d\xi^2 + d(V/\xi)/d\xi + [2\zeta R^2(1 + \nu)/E][\omega^2 V + 2\Omega\omega U] = 0 \quad (8)$$

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The boundary conditions (3) and (4) may now be written

$$dU/d\xi + (\nu/\xi)U = 0 \qquad \text{at } \xi = 1 \quad (9)$$

and

$$dV/d\xi - V/\xi = 0 \qquad \text{at } \xi = 1 \quad (10)$$

At this point, it is especially convenient to introduce a finite Hankel transform[4] defined as

$$\bar{f}(s) = \int_0^1 f(\xi)\xi J_n(s\xi)d\xi \quad (11)$$

where s is a positive root of the equation

$$sdJ_n(s)/ds + mJ_n(s) = 0 \quad (12)$$

J_n is a Bessel function of the first kind of order n, and m is a constant. It has been shown by Tranter[4] that the finite Hankel transform of the function $F(f)$ given by

$$F(f) = d^2f/d\xi^2 + d(f/\xi)/d\xi \quad (13)$$

is

$$\bar{F}(s) = J_1(s)[df/d\xi + mf]|_{\xi=1} - s^2\bar{f} \quad (14)$$

By using this transform in Eqs. (7) and (8), the following homogeneous algebraic equations are obtained for the transforms of U and V:

$$J_1(s)[dU/d\xi + (\nu/\xi)U]|_{\xi=1} - s^2\bar{U} + [\zeta R^2(1 - \nu^2)/E][\omega^2\bar{U} + 2\Omega\omega\bar{V}] = 0 \quad (15)$$

and

$$J_1(q)[dV/d\xi - V/\xi]|_{\xi=1} - q^2\bar{V} + [2\zeta(1 + \nu)R^2/E][\omega^2\bar{V} + 2\Omega\omega\bar{U}] = 0 \quad (16)$$

where s and q are positive roots of the equations

$$sdJ_1(s)/ds + \nu J_1(s) = 0 \quad (17)$$

and

$$qdJ_1(q)/dq - J_1(q) = 0 \quad (18)$$

The first terms in each of Eqs. (15) and (16) are zero because of the boundary conditions (9) and (10). Setting the determinant of the coefficients of the remaining terms equal to zero leads to the frequency equation

$$\omega_n^2 = 2\Omega^2 + (\tfrac{1}{2})(P_{un}^2 + P_{vn}) \pm [(\tfrac{1}{4})(P_{un}^2 - P_{vn}^2)^2 + 2\Omega^2(P_{un}^2 + P_{m}^2) + 4\Omega^4]^{1/2} \quad (19)$$

where P_{un} and P_{vn} are given by the expressions

$$P_{un}^2 = S_n^2E/[\zeta R^2(1 - \nu^2)] \quad (20)$$

and

$$P_{vn}^2 = q_n^2E/[2\zeta R^2(1 + \nu)] \quad (21)$$

where n is an index of the roots of Eqs. (17) and (18).

Equation (19) is identical in form to that predicted in Ref. 1. For small Ω^2 it may be written

$$\omega_n^2 \approx P_{un}^2 + 4\Omega^2P_{un}^2/(P_{un}^2 - P_{vn}^2) \quad (22)$$

and

$$\omega_n^2 \approx P_{vn}^2 - 4\Omega^2P_{vn}^2/(P_{un}^2 - P_{vn}^2) \quad (23)$$

As $\Omega^2 \to 0$, the frequencies become simply P_{un} and P_{vn}, which are natural in-plane vibration frequencies of a nonrotating disk.[3] Also, from these equations, it is shown that the rotation does produce a coupling between the two types of vibration and that the tangential (transverse) and radial (longitudinal) vibration frequencies are, respectively, decreased and increased.

References

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[3] Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, New York, 1927), 4th ed., pp. 496–498.

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Hypersonic Flow Visualization Using Tufts of Pure Carbon Yarn

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In hypersonic wind-tunnel experiments, tufts can be useful in a host of problems including visual assessment of spanwise flows in boundary layers on yawed and unyawed plane bodies, cross flows on pitched axisymmetric bodies, sting interference, open and closed wakes, and separated-reattaching flows. Tufts of wool, cotton, and similar organic materials have been used for some time in subsonic and in supersonic wind-tunnel experiments. However, the high temperatures associated with many hypersonic wind tunnels often have precluded the use of this simple yet extremely valuable technique. The purpose of this note is to suggest the use of tufts made of pure carbon yarn for high-temperature hypersonic wind-tunnel experiments.

Carbon tufts were used in hypersonic experiments at the Hypersonic Research Laboratory of the Aerospace Research Laboratories (U. S. Air Force) for flow visualization of hypersonic flows. These tufts exhibited many of the characteristics of wool. They were small and flexible, but they also had sufficient strength and temperature durability to withstand high temperatures associated with hypersonic tunnel flow conditions. Although a complete study of the characteristics and limitations of carbon yarn tufts has not been made, the following example should serve to demonstrate this point. The pictures in Fig. 1 show pure carbon yarn tufts, which were used on an axisymmetric model in a flow with a Mach number of about 14, a stagnation temperature of 2100° R, and a stagnation pressure of 1000 psia. The model's surface temperature was about 1300°R. The maximum model diameter is 0.75 in. In this case, the tufts were attached to the model with a very small amount of Pliobond (a suggestion of W. D. Humphries of Systems Research Corporation). The tufts shown contain as many as 50 strands. Tufts of from 2 to 5 strands also have been used with success under the same conditions, but photographs of these tufts were not suitable for reproduction. Tufts made of quartz fibers, Pyrex fibers, and a half-mil wire also were tried under similar conditions, but they were either too stiff or were attracted to the model's surface.

Questions as to the effects of these tufts on the flow field and the size and flexibility required to show the flow directions remain open. These are factors that depend upon each case being considered just as in lower speed flows. However, carbon yarn tufts are commercially available in a form that allows the experimentalist a wide latitude in selecting tuft lengths and thicknesses.

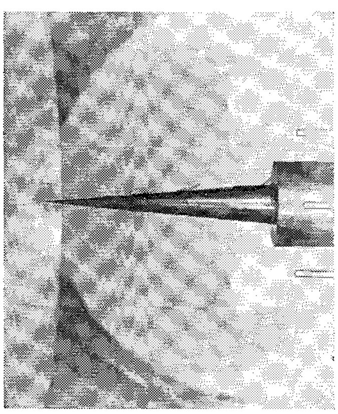

Fig. 1 Carbon yarn tufts in hypersonic separation experiment, Mach number 14, stagnation temperature 2100°R.

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Optimum Hypersonic Lifting Surfaces Close to Flat Plates

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Nomenclature

θ = wedge angle
$\tilde{p}_2$ = undisturbed wedge pressure
p' = dimensionless pressure perturbation
p_2 = $\tilde{p}_2(1 + p')$
M_2 = Mach number along unperturbed wedge
λ = reflection coefficient
k = $1 - [\tan\bar{\beta}/(M_2^2 - 1)^{1/2}]/1 + [\tan\bar{\beta}/(M_2^2 - 1)^{1/2}]$ = relative abscissa of wave and its reflection
$\bar{\beta}$ = angle between undisturbed shock wave and wedge
x = distance from apex along unperturbed wedge
γ = ratio of specific heats
Y_x = dimensional perturbation normal to original wedge surface

Studies of hypersonic lifting vehicles have led us to the investigation of two-dimensional lifting surfaces possessing maximum lift-to-drag ratio for fixed lift in the limit of inviscid hypersonic flow. Earlier work on this problem utilized hypersonic Newtonian theory to obtain two key results; the application of the slender-body version of Newtonian theory to the class of two-dimensional power law surfaces indicated that the flat plate is the optimum such surface in terms of $(C_L/C_D)_{\max}$ for fixed C_L, and the application of the full Newtonian-Busemann theory indicated that the optimum wing is a flat plate fitted with a narrow "Newtonian chine strip" at its trailing edge.[1]

Our approach to this problem is to consider the lower surface of a lifting airfoil, which is close to a flat plate in shape. By restricting our analysis to such shapes, we are able to use the first-order pressure correction caused by the perturbation of a wedge profile in supersonic flow, as given by Chernyi.[2]

The formula given by Chernyi (Ref. 2, p. 186), and quoted in this paper, stems from a first-order solution of the full equations of motion for supersonic flow of a perfect gas past a slightly perturbed wedge at arbitrary Mach number and wedge angle (Fig. 1).

The first-order pressure perturbation caused by the slight perturbation of a wedge profile in supersonic flow is

$$\tilde{p}_2 p'(x) = \frac{\gamma M_2^2 \tilde{p}_2}{(M_2^2 - 1)^{1/2}} \left[Y'(x) + 2 \sum_{n=1}^{\infty} \lambda^n Y'(k^n x) \right] \quad (1)$$

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